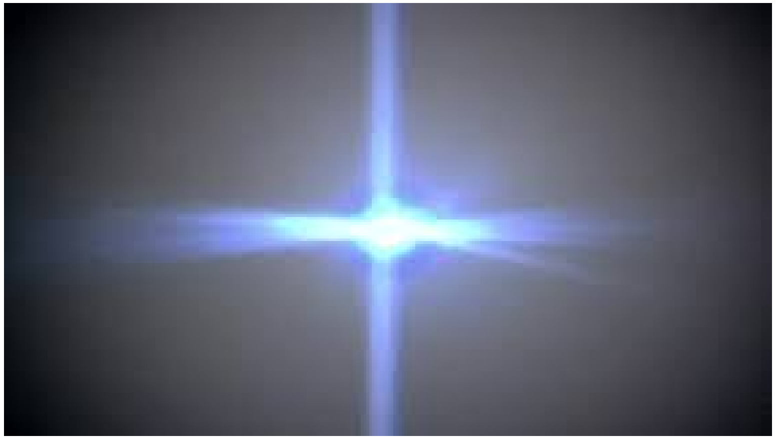
Do bubbles screen?

Karol Makuch

California Institute of Technology
Institute of Physical Chemistry of the Polish Academy of Sciences

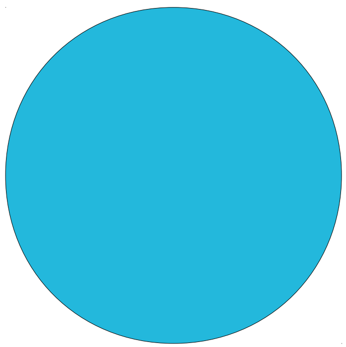
12.04.2017, Caltech

Introduction - bubbles



Often invisible... but crucial for understanding

Simplified description of liquids with bubbles



$$\dot{a} > 0$$

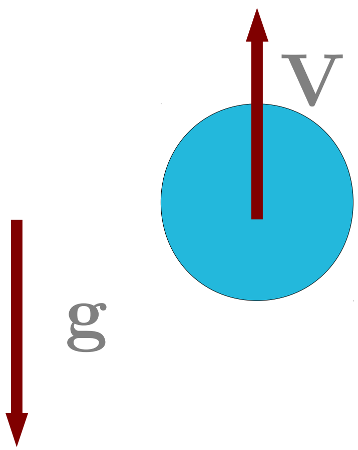
$$v(\mathbf{r}) \sim \frac{\hat{e}_r}{r^2}$$

The viscosity term in Navier-Stokes equations does not play a role, because

$$\Delta v(\mathbf{r}) = 0$$

The only effect of viscosity is force at the surface of bubble

Simplified description of liquids with bubbles



Moore [JFM, 1963, 16, 161-176]:
For high Reynolds number

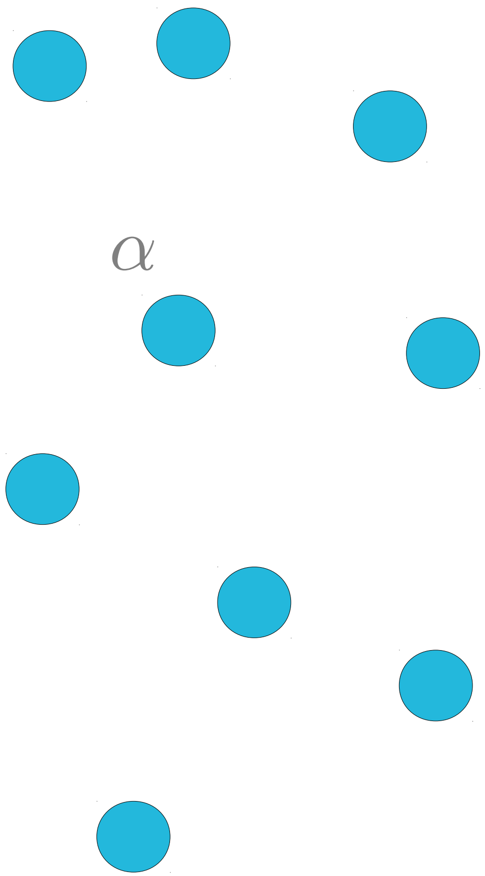
$$Re = \frac{aV}{\eta}$$

viscosity plays a role only in boundary layer around bubble. It produces drag force

$$F = 12\pi\mu aV$$

Flow is irrotational

Simplified description of liquids with bubbles



$\alpha = 1, \ldots, N$

$$\nabla \times \mathbf{v} = 0 \qquad \nabla \cdot \mathbf{v} = 0$$

$$\mathbf{v} = \nabla \phi$$

$$\Delta \phi = 0$$

$$\hat{n}(\mathbf{r}) \cdot \nabla \phi \big|_{\mathbf{r} \in \partial \Omega^\alpha} = \dot{a}_\alpha + \hat{n}(\mathbf{r}) \cdot \dot{\mathbf{R}}^\alpha$$

$$= \sum_{i=0}^{3} n_i(\mathbf{r}) \dot{X}_i^\alpha$$

$$X_0^\alpha = a_\alpha \qquad X_{i=1,2,3}^\alpha = \mathbf{R}_i^\alpha$$

$$n_0 = 1 \qquad n_{i=1,2,3} = \hat{n}_i(\mathbf{r})$$

Solution of Laplace equation

$$\Delta\phi = 0$$

$$\hat{n}(\mathbf{r}) \cdot \nabla\phi|_{\mathbf{r}\in\partial\Omega^{\alpha}} = \sum_{i=0}^{3} n_i(\mathbf{r})\dot{X}_i^{\alpha}$$

Solution of the above problem is linear in velocities

$$\phi(\mathbf{r}, X, \dot{X}) = \sum_{\beta=1}^{N} \Phi_j^{\beta}(\mathbf{r}, X)\dot{X}_j^{\beta}$$

Kinetic energy

$$T = \int_{\Omega_{\mathrm{fluid}}} d^3 r \, \rho \frac{\mathbf{v}^2 \left(\mathbf{r}\right)}{2}$$

Using Stokes theorem and velocity field by potential:

$$T = \frac{1}{2} \dot{X}_i^{\nu} M_{ij}^{\nu\mu} \left(X\right) \dot{X}_j^{\mu}$$

with virtual mass matrix given as follows:

$$M_{ij}^{\alpha\beta} \left(X\right) = -\rho \int_{\partial\Omega_\alpha} d^2 r \, n_i(\mathbf{r}) \Phi_j^{\beta} \left(\mathbf{r}; X\right)$$

Dynamics of bubbly liquid

Lagrange equations with hydrodynamic force

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{X}_i^\alpha}\right) - \frac{\partial T}{\partial X_i^\alpha} = \int_{\partial\Omega_\alpha} d^2r\, n_i(\mathbf{r})p(\mathbf{r})$$

With explicit expression for kinetic energy by virtual mass matrix:

$$\int_{\partial\Omega_\alpha} d^2r\, n_i(\mathbf{r})p(\mathbf{r}) =$$

$$= M_{ij}^{\alpha\beta}(X)\ddot{X}_j^\beta + \underline{\dot{X}_k^\gamma \frac{\partial}{\partial X_k^\gamma} M_{ij}^{\alpha\beta}(X)\dot{X}_j^\beta - \frac{1}{2}\dot{X}_k^\gamma \dot{X}_j^\beta \frac{\partial}{\partial X_i^\alpha} M_{kj}^{\gamma\beta}(X)}$$

Proof for bubbly liquids with bubbles which does not change their radius
(e.g. Hinch, E. & Nitsche, L. C. JFM, 1993, 256, 343-401)

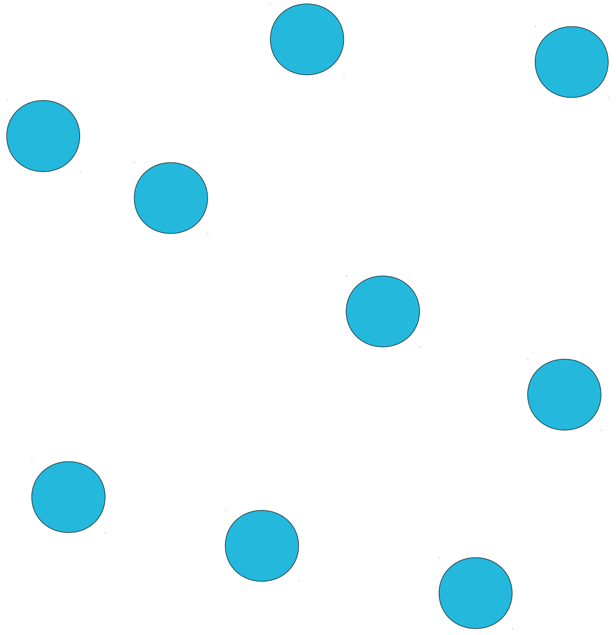
In bubbly liquids mass depends on position (virtual mass matrix).

Virtual mass matrix

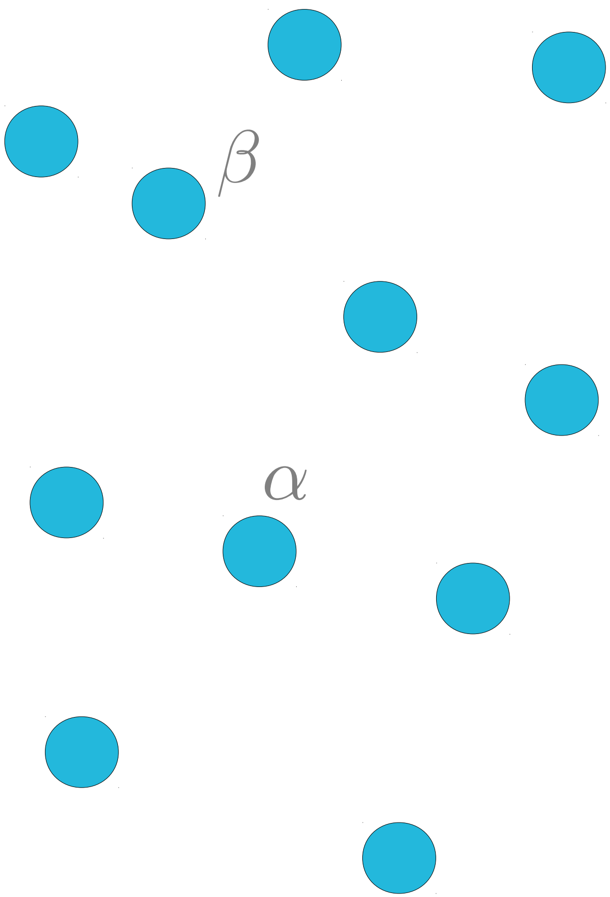
$$T = \frac{1}{2} \dot{X}_i^{\nu} M_{ij}^{\nu\mu} (X) \dot{X}_j^{\mu}$$

- Determines dynamic of the system
- Depends on configuration of all bubbles in liquid

What are properties of virtual mass matrix?



“Simple” problem



Quiescent fluid, no motion

$$\dot{X} = 0$$

… and pressure (or force) $\mathcal{F}^{\beta}$ appears suddenly on the bubble β .

What is acceleration of bubbles $\ddot{X}^{\alpha}$?

The answer follow from equation of motion:

$$\ddot{X}^{\alpha} = [M^{-1}(X)]^{\alpha\beta}\mathcal{F}^{\beta}$$

How it depends on the distance between bubbles? - essential for formulation of effective equations of bubbly liquid

How to calculate virtual mass matrix?

Laplace equation for potential of the velocity field with boundary conditions on the surfaces of bubbles

$$\Delta\phi = 0$$

$$\hat{n}(\mathbf{r}) \cdot \nabla\phi\big|_{\mathbf{r}\in\partial\Omega^{\alpha}} = \theta^{\alpha}\left(\hat{n}^{\alpha}\right) \equiv \dot{a}_{\alpha} + \hat{n}(\mathbf{r}) \cdot \dot{\mathbf{R}}^{\alpha}$$

Integral representation of the equation:

$$\phi\left(\mathbf{r}\right) = \int d^3r'\, G\left(\mathbf{r}-\mathbf{r}'\right) s\left(\mathbf{r}'\right)$$

With Green function of Laplace equation:

$$G\left(\mathbf{r}\right) = \frac{1}{4\pi r}$$

And source terms on the surfaces of particles:

$$s\left(\mathbf{r}\right) = \sum_{\alpha=1}^{N} s^{\alpha}\left(\frac{\mathbf{r}-\mathbf{R}^{\alpha}}{\left|\mathbf{r}-\mathbf{R}^{\alpha}\right|}\right)\delta\left(\left|\mathbf{r}-\mathbf{R}^{\alpha}\right| - a^{\alpha}\right)$$

Multipole expansion

Normal component of the velocity on the surface of the bubble by multipole expansion:

$$\theta^{\alpha}(\hat{n}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \Theta_{lm}^{\alpha} Y_{lm}(\hat{n})$$

Normal component of the source on the surface of the bubble by multipole expansion:

$$s^{\alpha}(\hat{n}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} S_{lm}^{\alpha} Y_{lm}(\hat{n})$$

Virtual mass matrix

$$M = Z_0 + Z_< G \left[1 - \hat{Z} G\right]^{-1} Z_>$$

$$\hat{Z}_{\alpha l m \beta l' m}(X^\alpha) = \delta_{\alpha\beta}\delta_{ll'}\delta_{mm'}\frac{1}{a_\alpha}\frac{(2l+1)l}{l+1}$$

$$G_{\alpha l m, \beta l' m'}(X^\alpha, X^\beta) =$$
$$= \int d\hat{n}^\alpha \int d\hat{n}^\beta Y^*_{lm}(\hat{n}^\alpha) G(\hat{n}^\alpha a^\alpha + \mathbf{R}^\alpha - \mathbf{R}^\beta - \hat{n}^\beta a^\beta) Y_{l'm'}\left(\hat{n}^\beta\right) a_\beta^2$$

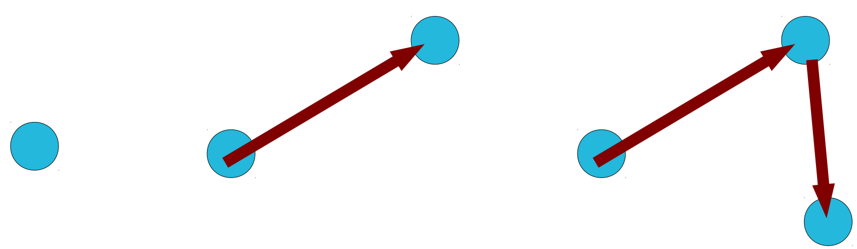
$$\sim \frac{1}{\left|\mathbf{R}^\alpha - \mathbf{R}^\beta\right|^{l+l'+1}}$$ for $\left|\mathbf{R}^\alpha - \mathbf{R}^\beta\right| \to \infty$

Virtual mass matrix – scattering series interpretation

$$M = Z_0 + Z_< G \left[1 - \hat{Z} G\right]^{-1} Z_>$$

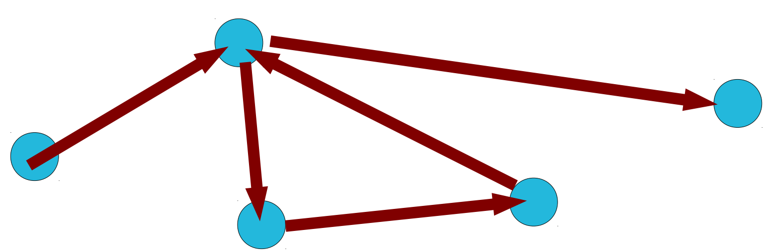
Using geometric series expansion

$$M = Z_0 + Z_< G Z_> + Z_< G \hat{Z} G Z_> + \dots$$



All possible paths in general:

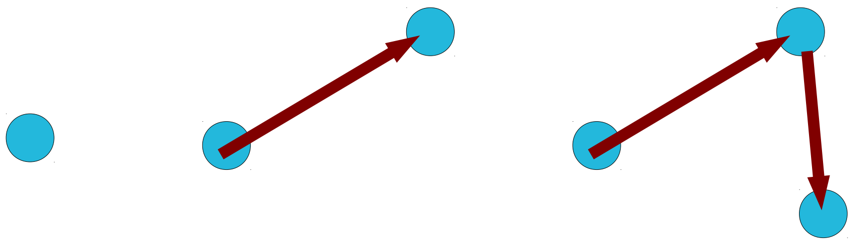
many-body “interactions”



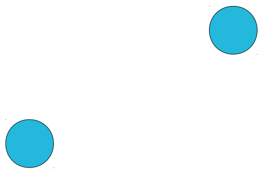
Inverse of virtual mass matrix

$$M = Z_0 + Z_< G \left[1 - \hat{Z} G\right]^{-1} Z_>$$

$$M^{-1} = \zeta_0 + \zeta_< G \left[1 - \hat{\zeta} G\right]^{-1} \zeta_>$$



Virtual mass matrix – example



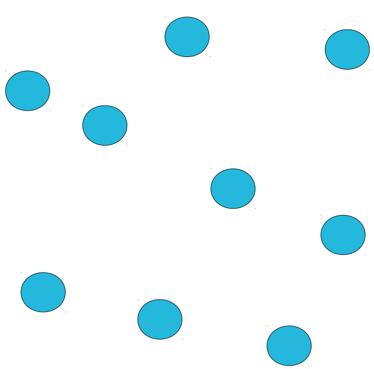
$$\dot{\mathbf{R}}^1 = 0 \qquad \dot{\mathbf{R}}^2 = 0$$

$$\dot{a}_1 \qquad \dot{a}_1$$

Ilinskii, Y. A.; Hamilton, M. F. & Zabolotskaya, E. A. JASA, 2007, 121, 786-795

Kinetic energy – dominant contribution (valid for distant droplets)

$$T = \frac{1}{2}\dot{a}_1^2 4\pi\rho a_1^3 + \frac{1}{2}\dot{a}_2^2 4\pi\rho a_2^3 + \frac{1}{2}4\pi\rho\frac{a_1^2 a_2^2}{|\mathbf{R}^1 - \mathbf{R}^2|}\dot{a}_1\dot{a}_2$$



Clouds of bubbles in two-body approximation:

$$T \ni \sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}\frac{1}{2}4\pi\rho\frac{a_\alpha^2 a_\beta^2}{|\mathbf{R}^\alpha - \mathbf{R}^\beta|}\dot{a}_\alpha\dot{a}_\beta$$

For homogeneous infinite cloud sum is infinite → strong collective effects? → behavior depends on the shape of the cloud, even if the boundary is in “infinity”

Virtual mass matrix – two-body approximation

Ilinskii, Y. A.; Hamilton, M. F. & Zabolotskaya, E. A. JASA, 2007, 121, 786-795:
Interaction of close bubbles is not important, because:

the much smaller quantity R/R_{cl}. The physical explanation for this effect is that for a uniform distribution of bubbles, those farther away collectively exert greater influence because their number per unit volume increases as r^2, which more than compensates for the influence of an individual bubble, whose effect decreases as $1/r$. For N sufficiently large, Eqs. (35) and (37) therefore remain valid even when R/d is of order one for isolated pairs of bubbles.

A similar argument can be made using the kinetic energy, and it therefore applies also to the Hamiltonian equations developed in Sec. IV.

Langrangian:

$$L(X,\dot{X}) = \frac{1}{2}\dot{X}M(X)\dot{X}$$

Hamiltonian (momenta)

$$P = M(X)\dot{X}$$

$$H(X,P) = \frac{1}{2}PM^{-1}(X)P$$

What about many-body interactions? Any collective effects?

To sum many-body long-range interactions in virtual mass matrix - fluctuation expansion

$$M = Z_0 + Z_< G Z_> + Z_< G \hat{Z} G Z_> + \dots$$

Before introducing fluctuation expansion – let's use the following representation

$$\langle [\dots] \rangle \equiv \int d^3R^1 \dots d^3R^N [\dots] p(\mathbf{R}^1, \dots, \mathbf{R}^N)$$

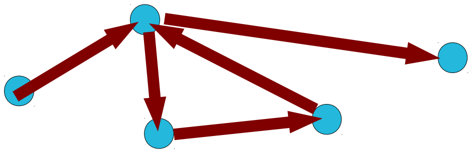
$$M^{\alpha\beta}(X) \to M(\mathbf{R}, \mathbf{R}'; X) \equiv \sum_{\alpha=1}^{N} \sum_{\beta=1}^{N} \delta(\mathbf{R} - \mathbf{R}^\alpha) M^{\alpha\beta}(X) \delta(\mathbf{R}' - \mathbf{R}^\beta)$$

$$Z^{\alpha\beta}(X) \to Z(\mathbf{R}, \mathbf{R}'; X) \equiv \sum_{\alpha=1}^{N} \delta(\mathbf{R} - \mathbf{R}^\alpha) \hat{Z}^{\alpha\alpha}(X) \delta(\mathbf{R}' - \mathbf{R}^\alpha)$$

$$M = Z_0 + Z_< G Z_> + Z_< G \hat{Z} G Z_> + \dots$$

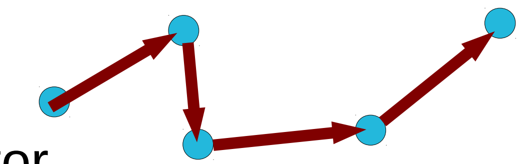
Fluctuation expansion

$$G[1-\hat{Z}G]^{-1} = G\left[1-\left(\hat{Z}-\langle\hat{Z}\rangle+\langle\hat{Z}\rangle\right)G\right]^{-1}$$



$$(1+a+b)^{-1} = (1+a)^{-1}\left[1+b(1+a)^{-1}\right]^{-1}$$

$$= G_{\langle\hat{Z}\rangle}\left[1-\left(\hat{Z}-\langle\hat{Z}\rangle\right)G_{\langle\hat{Z}\rangle}\right]^{-1}$$



effective propagator

fluctuation

$$G_{\langle\hat{Z}\rangle} \equiv G\left[1-\langle\hat{Z}\rangle G\right]^{-1}$$

$$\left\langle\left(\hat{Z}-\langle\hat{Z}\rangle\right)G_{\langle\hat{Z}\rangle}\left(\hat{Z}-\langle\hat{Z}\rangle\right)\right\rangle(\mathbf{R},\mathbf{R}') \to n^2 h(\mathbf{R},\mathbf{R}')G_{\langle\hat{Z}\rangle}(\mathbf{R},\mathbf{R}')$$

As a result of resummation: propagation is from fluctuation to fluctuation with effective propagator.

Important observation: asymptotics of virtual mass matrix

$$\langle M\rangle(\mathbf{R}-\mathbf{R}') \sim G_{\langle\hat{Z}\rangle}(\mathbf{R}-\mathbf{R}')$$

Asymptotics of inverse of mass matrix

$$\left\langle M^{-1}\right\rangle(\mathbf{R}-\mathbf{R}') \sim G_{\langle\hat{\zeta}\rangle}(\mathbf{R}-\mathbf{R}')$$

From multipole to cartesian and to Fourier space:

$$\hat{G}_{\langle\hat{\zeta}\rangle}(\mathbf{k}) = \hat{G}(\mathbf{k})\left[1-\langle\hat{\zeta}\rangle(\mathbf{k})\hat{G}(\mathbf{k})\right]^{-1}$$

$$\hat{G}(\mathbf{k}) = \frac{1}{k^2}$$

No matrices, only scalars

$$\langle\hat{\zeta}\rangle(\mathbf{k}) = 4\pi an\left[-1+\frac{(ak)^4}{27}\right]+O((ak)^6), \qquad k\to 0$$

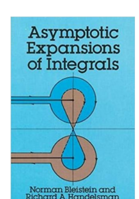
$$\hat{G}_{\langle\hat{\zeta}\rangle}(\mathbf{k}) = \frac{1}{k^2+4\pi an}+O(k^2)$$

Only even powers of k

All p:

$$G_{\langle\hat{\zeta}\rangle}(\mathbf{r}) = O(r^{-p}) \qquad r\to\infty$$

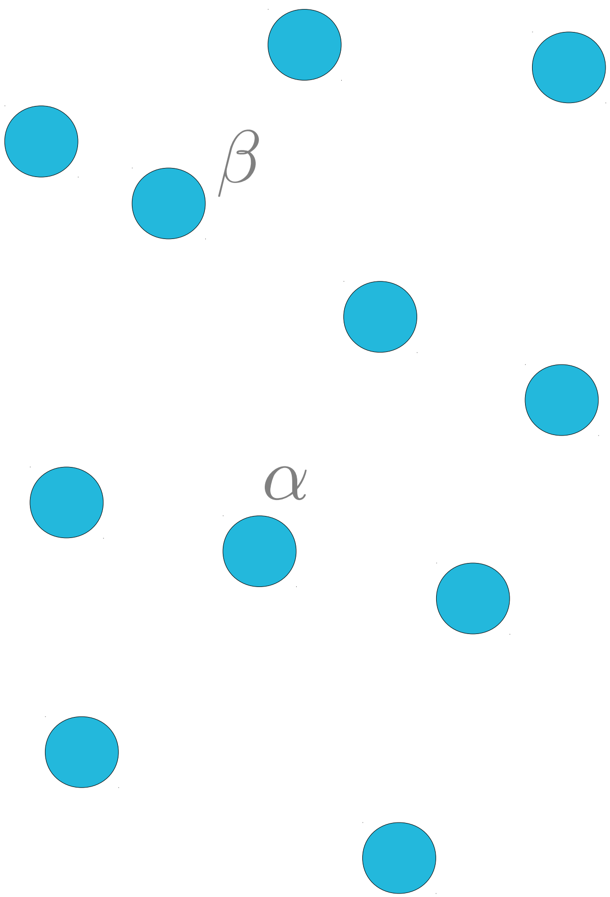
Screened Coulomb potential



Result

$$\ddot{X}^{\alpha} = [M^{-1}(X)]^{\alpha\beta}\mathcal{F}^{\beta}$$

$$\dot{X} = 0$$



How it depends on the distance between bubbles?

In two body approximation:

$$[M^{-1}(X)]^{\alpha\beta} \sim \frac{1}{|\mathbf{R}^{\alpha} - \mathbf{R}^{\beta}|}$$

But with collective effects:

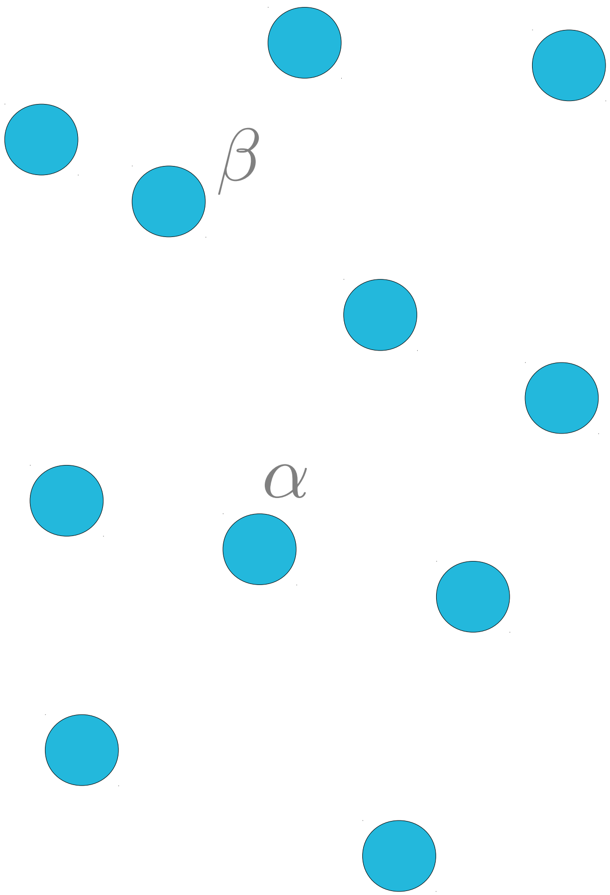
$$[M^{-1}(X)]^{\alpha\beta} \sim \frac{e^{-|\mathbf{R}^{\alpha} - \mathbf{R}^{\beta}|}}{|\mathbf{R}^{\alpha} - \mathbf{R}^{\beta}|}$$

Mechanical screening in bubbly liquids!

Surprising difference between compressible and incompressible bubbles in bubbly liquid

$$\dot{\mathbf{R}}^{\gamma} = 0$$

$$\ddot{\mathbf{R}}^{\alpha} = [M^{-1}(X)]^{\alpha\beta} F^{\beta}$$



How it depends on the distance between bubbles?

In two body approximation:

$$[M^{-1}(X)]^{\alpha\beta} \sim \frac{1}{|\mathbf{R}^{\alpha} - \mathbf{R}^{\beta}|^{3}}$$

It is the same behavior when collective effects are taken into account.

No mechanical screening in bubbly liquids with incompressible bubbles

Summary

Bubbles in bubbly liquids influence their mutual motion.

The information about the above interactions is in the virtual mass matrix.

$$H(X,P) = \frac{1}{2} P M^{-1}(X)\, P$$

Inverse of virtual mass matrix is strongly influenced by collective effects of bubbles – mechanical screening:

Two body approximation:

$$[M^{-1}(X)]^{\alpha\beta} \sim \frac{1}{|\mathbf{R}^{\alpha} - \mathbf{R}^{\beta}|}$$

Rigorous result for cloud of bubbles:

$$[M^{-1}(X)]^{\alpha\beta} \sim \frac{e^{-|\mathbf{R}^{\alpha} - \mathbf{R}^{\beta}|}}{|\mathbf{R}^{\alpha} - \mathbf{R}^{\beta}|}$$

↓

Consequences for effective equations describing bubbly liquids?